A Systematic Approach to Derivations

A great way to approach derivations is to work from both the top and bottom. This allows you to keep an eye on both where you need to go (i.e. the formulas you must derive) and the tools you have to get there (i.e. the formulas you have to derive with).

At each stage in the derivation, you should ask yourself the following questions about the formula you want to derive:

1. Can I Extract It?

Can I extract the formula from the formulas I have—the accessible formulas? If the formula you want to extract appears as the consequent of an accessible conditional formula (conditional-elimination), appears as one of the conjuncts of an accessible conjunction (conjunction-elimination), can be derived from both disjuncts of an accessible disjunction (disjunction-elimination), or appears on either side of an accessible biconditional (biconditional-elimination), then the answer is *yes*. If it doesn't, then the answer's *no*.

So, for instance, if the only accessible formula is $(p \supset q)$ and you're wanting to arrive at $(\sim p \vee q)$, then it's not clear you'll be able to extract this formula from what you have. If the accessible formulas above are $(p \supset (\sim p \vee q))$ and p, however, you're in business. It'll be easy to get $(\sim p \vee q)$.

2. Can I Build It?

Assuming you can't extract the formula you want, you should then ask yourself: can I *build* the formula, at least in principle? In other words, can I use disjunction-introduction, conjunction-introduction, conditional-introduction, biconditional-introduction, or negation-introduction to build the formula from smaller parts?

If the formula's main connective is a $\supset$, $\vee$, $\equiv$, &, or ~, then the answer is: yes.

If the formula is atomic, then the answer is: no.

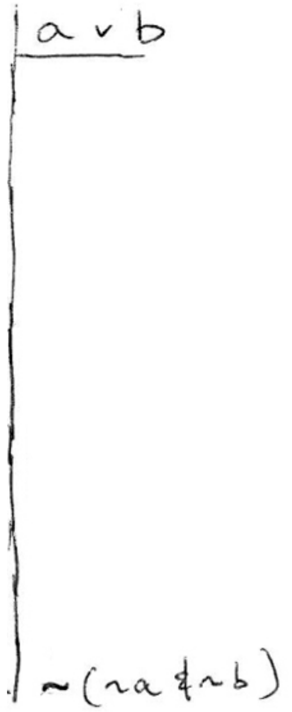
Assuming you can build the formula in principle, you now have to see whether you can do so *in practice*. Look to see whether the formulas you have to work with are ones with which you can build the formula you want. For instance, if the only formula above is $\sim(p \supset \sim q)$ and you're hoping to build $(p \,\&\, q)$, then building looks foreclosed; it's not clear how you'll be able to get p and q with which to build $(p \,\&\, q)$. If, however, the formulas above are $(p \supset q)$ and p, then you're in business; it'll be easy to get p and q (and thus, $(p \,\&\, q)$).

3. If in Doubt, Eliminate that Negation

Assuming you can neither extract nor build the formula you want, your only remaining option is to derive it by contradiction using negation-elimination. If you want formula A, then simply begin a subderivation with assumption ~A and try to derive a contradiction.

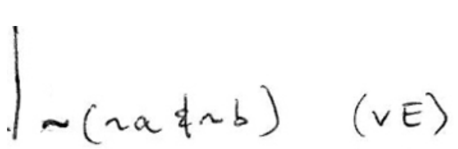
Putting the Method to Work

Let's say we want to show the following: *a* ∨ *b* $\vdash_D$ ~(~*a* & ~*b*). How should we start? Using the method described, we want to write down both what we have to work with at the top of our derivation, and what we aim to derive at the bottom. So we write the following:



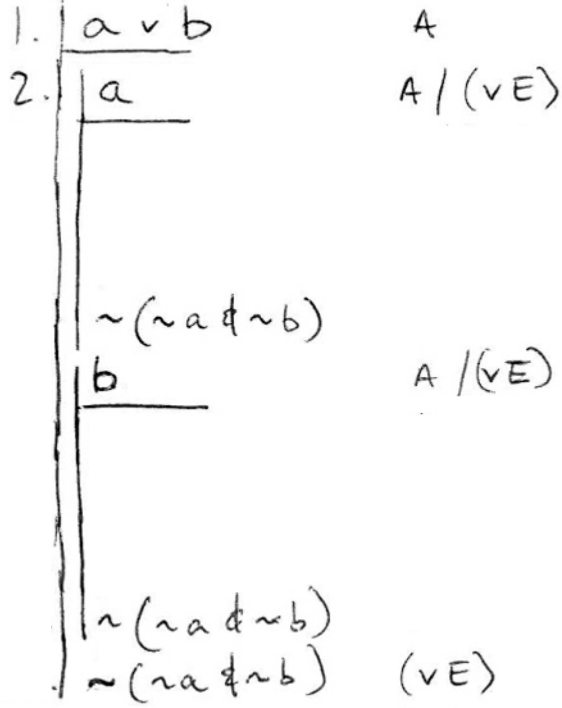
Now we must ask our questions about the formula we wish to derive, namely: *~(~a & ~b)*. First, can we extract the formula? You might think not, since the formula doesn't appear embedded anywhere above. However, notice that the formula we *do* have to work with (namely, *(a ∨ b)*) is a disjunction. This means we can, in principle, get our formula by disjunction-elimination by deriving *~(~a & ~b)* from *a* and from *b*, respectively. Indeed, whenever we have a disjunction as a premise, it's a good idea to see if we can use it to get to what we want. Notice also that just in terms of truth-values *a* clearly entails *~(~a & ~b)*, as does *b*. This tells us that, at the very least, trying to derive *~(~a & ~b)* from *a* or from *b* isn't doomed to failure. So let's try disjunction-elimination:

First, we'll want to write the disjunction-elimination justification beside the formula we're deriving in this way:



This is just provisional. If our approach doesn't work, there may be others (like negation-introduction) that will. In that case, we'd start again, deleting this justification and entering a new one. But, for now, we'll assume our approach will work. Moreover, writing a provisional justification helps us remember the approach we're currently pursuing.

What do we need for disjunction-elimination? Well, we need a disjunction A ∨ B and then two subderivations: one beginning with A and ending in the formula we want, call it C, and another beginning with B and ending in C. Given that our disjunction is *a* ∨ *b* and the formula we want to derive is *~(~a & ~b)*, we'll want a subderivation that begins with *a* and ends in *~(~a & ~b)* and another subderivation that begins with *b* and ends in *~(~a & ~b)*. We can add these two subderivations to our derivation as follows (we'll worry about how to get from *a* to *~(~a & ~b)* and *b* to *~(~a & ~b)* in the next step):

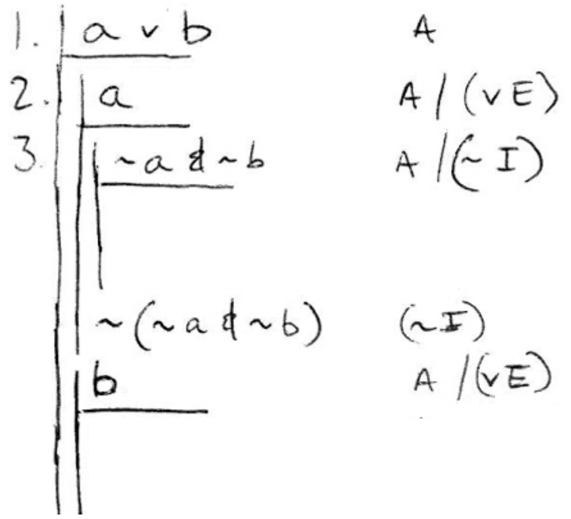


As you can see, we also write 'A/(∨E)' beside the two assumptions, *a* and *b*, to remind us why we have made these assumptions.

Now we have two subderivations starting with *a* and *b*, respectively, and both ending in *~(~a & ~b)*. But it's not obvious how to get from each assumption to the end. Let's focus on the first subderivation—the one beginning with *a* at line 2.

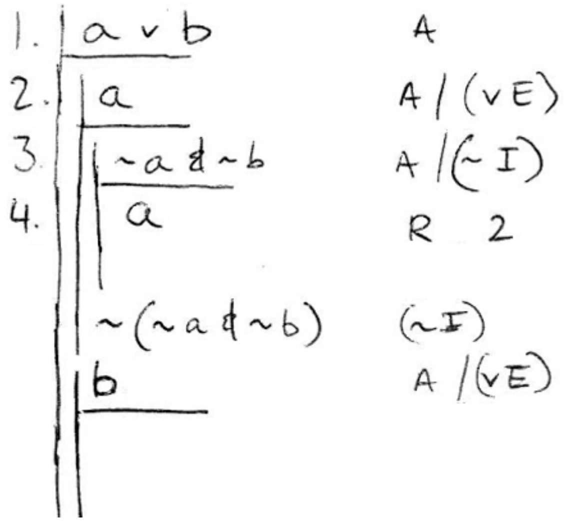
How do we get *~(~a & ~b)* in this subderivation? We must ask our questions above again: (1) Can we extract *~(~a & ~b)*? Well, it's the same formula we asked this about last time. So, again, the answer's *yes*. We could extract the formula by the same method again, disjunction-elimination, using the disjunction from line 1. But, if we think about it for a second, we'll see that we shouldn't repeat this strategy at this stage, since otherwise we'll just go on indefinitely starting nested subderivations beginning with *a* and *b* and ending in *~(~a & ~b)* and never getting anywhere. Since there's no other way to extract *~(~a & ~b)* from the accessible formulas, *a* and *(a ∨ b)*, we must ask, can we *build* it? The answer to this is also *yes*. Since the main connective of *~(~a & ~b)* is ~, we can build the formula by negation-introduction. So, let's try this.

We know that negation-introduction requires a subderivation that starts with an assumed formula A and ends in B, ~B (formulas that contradict), allowing us to introduce ~A on the scope-line to the left. As such, if we want to derive *~(~a & ~b)* by negation-introduction, we'll need to start another subderivation above it beginning with *(~a & ~b)*, besides which we write 'A/(~I)', to remind us why we made this assumption. We also need to write '(~I)' as a justification beside *~(~a & ~b)* following this subderivation. Again, this justification is provisional—contingent on our approach working out:



What we need now then, in order to use negation-introduction, is to derive two formulas that contradict within this subderivation. Whenever we're faced with this task, we must simply look above at the accessible formulas and see how to get a contradiction out of them.

There's no obvious way to get a contradiction with *(a ∨ b)* (line 1), likewise *(~a ∨ ~b)* (line 3). That leaves us with only one other option at this stage: derive a contradiction with *a* (line 2)—namely, *~a*. So first, we reiterate *a*, writing 'R 2' beside line 4.



So, now we need *~a* to get our contradiction. So we ask our questions again: (1) can we extract *~a*? Yes we can; notice that we have available to us *(~a & ~b)* on line 3. Using conjunction-elimination, we can very easily get *~a*. So let's do just that:

1.	$a \lor b$	A
2.	a	A/($\lor$E)
3.	$\sim a \,\&\, \sim b$	A/($\sim$I)
4.	a	R 2
5.	$\sim a$	(&E) 3
6.	$\sim(\sim a \,\&\, \sim b)$	($\sim$I) 3-5
7.	b	A/($\lor$E)
	$\sim(\sim a \,\&\, \sim b)$	
	$\sim(\sim a \,\&\, \sim b)$	($\lor$E)

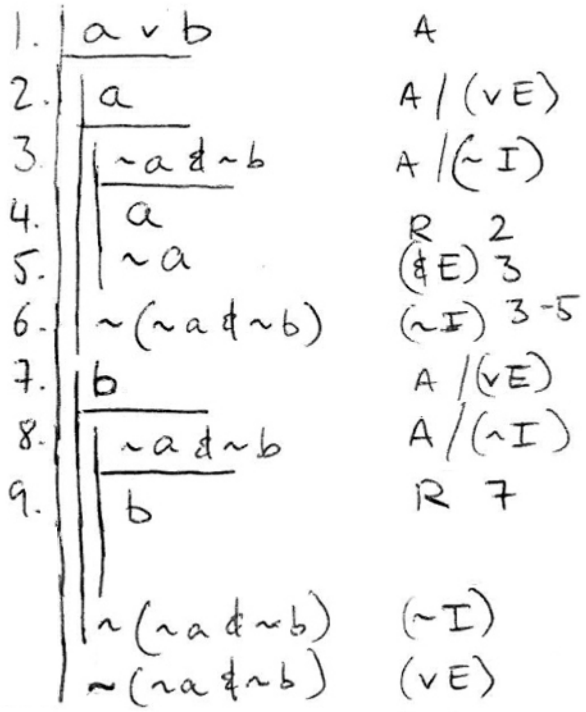
We write '&E 3' beside the line to show that we performed conjunction-elimination from line 3. We write '3-5' beside line 6 as these lines encompass the subderivation that justifies the formula on this line.

This completes the first subderivation needed for our grander project of disjunction-elimination. It shows that *~(~a & ~b)* follows from *a*. We must now complete the second subderivation to show that *~(~a & ~b)* also follows from *b*. Having done this we will be able to derive *~(~a & ~b)* by disjunction-elimination.

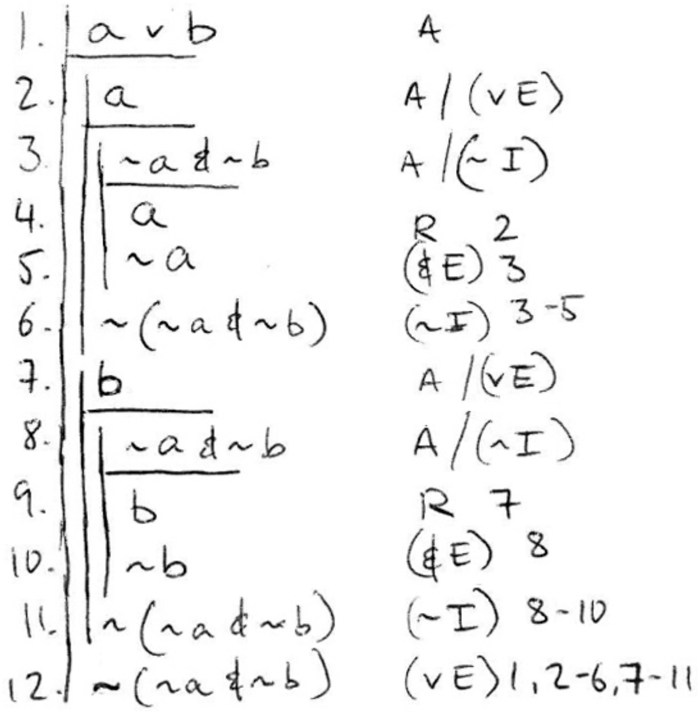
Completing the second subderivation proceeds exactly as the first did. First we begin a subderivation with *(~a & ~b)* with the intention of getting *~(~a & ~b)* by negation-introduction, writing 'A/(~I)' and '(~I)' beside the appropriate lines.

7.	b	A/($\lor$E)
8.	$\sim a \,\&\, \sim b$	A/($\sim$I)
	$\sim(\sim a \,\&\, \sim b)$	($\sim$I)
	$\sim(\sim a \,\&\, \sim b)$	($\lor$E)

Now we need a contradiction as before. We get it in almost identical fashion, by first reiterating *b* from line 7 and writing 'R 7' in the margin:



We then extract *~b* by conjunction-elimination from line 8, just as we extracted *~a* in the previous subderivation from line 3. We write '(&E) 8' in the margin to justify the new line. Finally, we write the lines needed to justify *~(~a & ~b)* on the penultimate line—namely, '8-10'. We also do this for the very last line. Here we write '1', since this is the disjunction we used for disjunction-elimination; '2-6', as this encompasses the first necessary subderivation; and '7-11', as this encompasses the second necessary subderivation.



This completes the derivation.